ECON 352 - Macroeconomics

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Abstract

These notes are heavily based on Professor Francisco Alvarez-Cuadrado's notes and lectures for ECON 352D1

1 Mathematical Tools

This year, we've added differential equations to our math toolkit in Economics. Luckily for us, those we are dealing with are rather simple to solve.

1.1 Notation

- $\dot{y}$: Denotes us the partial derivative of y relative to time.
- y_x: Denotes the partial derivative of y relative to x.
- $\hat{y} = \frac{\dot{y}}{y}$: Denotes the growth rate of y relative to time.

1.2 Differential equations

A **differential equation** is simply an equation that involves a derivative. To some extent, we've already dealt with them in earlier Calculus courses when we did implicit differentiation. An example would be $5y(t) - 2y'(t) + 4 = 7$ where we call t the **independent variable** and $y(t)$ the **dependent variable**. As opposed to regular equations, the solution of a differential equation is a function $y(t)$ of the independent variable and not a specific value.

Autonomous differential equation: A differential equation where the independent variable only enters the equation through the dependent variable. The example above is autonomous while $5y(t) - 2'y(t) + 4t = 7$ is not.

Generally, differential equations arise when we model real-life phenomena where the **rate of change** (i.e. the derivative) of some variable is in some way affected by the **current level** of the variable. They are particularly useful in Economics since once we solve them, we get a function that describes for us the value of a variable of our model at any point in time.

1.3 Solving Differential Equations

There are two ways to solve differential equations: **analytically** meaning that we use Calculus to find the function that fits the equation or **qualitatively** by drawing a phase diagram that represents the equation.

1.3.1 Analytically

In this course we will only be analytically solving **first-order differential equations** (meaning that the highest degree derivatives we deal with are first degree). These equations have a general form given by:

$$\dot{y}(t) + u(t)y(t) = w(t)$$

These equations can be either **homogeneous** (meaning the right-hand side (from now on abbreviated RHS) is 0) or **non-homogeneous** (meaning the RHS is non-zero). While the notes only consider the case where $u(t)$ and $w(t)$ are constants, here is the derivation of the general solution for the more general case.

Homogeneous case: $\dot{y}(t) + u(t)y(t) = 0$. Since $\dot{y}(t)$ is just the derivative of y with respect to time, it can be written as $\frac{dy(t)}{dt}$. Then, by bringing $u(t)y(t)$ to the RHS we get:

$$\frac{dy(t)}{dt} = -u(t)$$

Since dt represents a very small change in time, you can multiply both sides by it to get:

$$dy(t) = -u(t)y(t)dt$$

We then divide both sides by $y(t)$ and integrate both sides to get:

$$\int \frac{1}{y(t)} dy(t) = \int -u(t)dt$$

Since the integral of $\frac{1}{x}$ is simply $ln(x)$, we get:

$$ln(y(t)) + c_1 = -\int u(t)dt$$

Exponentiating both sides yields:

$$y(t) + e^{c_1} = e^{-\int u(t)dt}$$

And since e^{c_1} is just some constant C_1, we have that what we call the **general solution** of the equation is given by:

$$y(t) = C_1 e^{-\int u(t)dt}$$

In the case where $u(t)$ is just some constant u, we have that $-\int u(t)dt = -\int udt = -ut + c$, meaning that the general solution is given by:

$$y(t) = C_1 e^{-ut+c} = Ce^{-ut}$$

Where e^c is multipled by the previous constant to give a new one.

Non-Homogeneous case: $\dot{y}(t) + uy(t) = w$. When the equation is non-homogeneous, the solution is given by the sum of the **complementary solution** and the **particular solution**.

- **Complementary solution**: Given by the **general solution** of the homogeneous case ($y(t) = Ce^{-ut}$)
- **Particular solution**: Given by any function that solves the complete equation. We consider the case where $y(t) = k$ where k is a constant. Then, we get that $\dot{y} = 0$ and plugging into the equation gives us:

$$0 + uk = w \iff k = \frac{w}{u} \implies y(t) = \frac{w}{u}$$

The general solution is thus:

$$y(t) = Ce^{-ut} + \frac{w}{u}$$

Remark: In the case where $u = 0$, we can just apply the homogeneous case by noticing $\dot{y} = w \iff \dot{y} - w = 0$

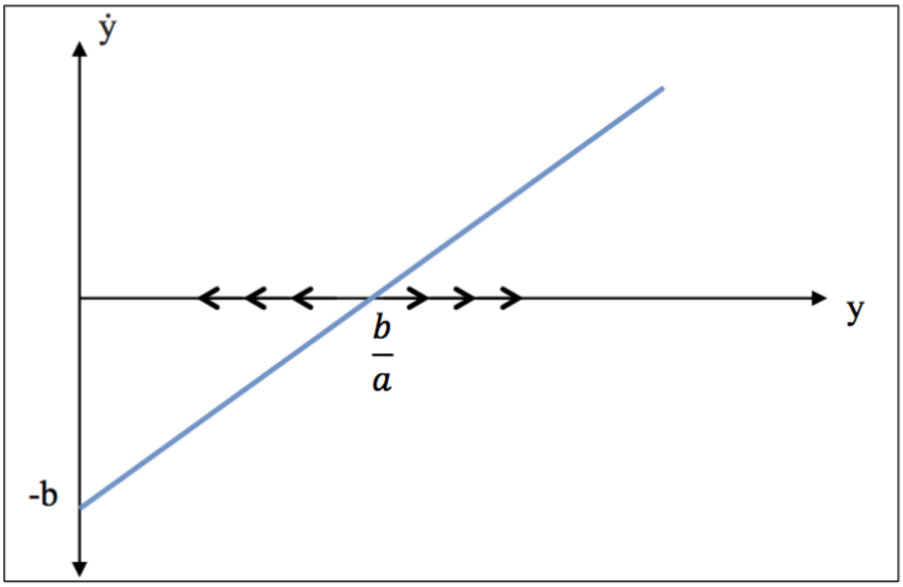
The only thing left is to determine the value of C. It is given by using an initial condition (ex: $y(0) = 10$) and solving for C. In the example, $y(0) = Ce^{-u0} + \frac{w}{u} \implies C = y(0) - \frac{w}{u}$

1.3.2 Qualitatively

Solving differential equations can also be done qualitatively in cases where it is hard for us to get an analytical answer. This process is done by drawing a graph that relates the level of our variable to its rate of change (a **phase diagram**). Once the graph is drawn, we find the **steady state** (when the level of our variable is not changing) by setting the rate of change equal to 0 and solving for the level. Once that's done, we draw arrows around the steady state to indicate how the level of the variable varies if it is changed from its steady state.

For example, let's consider the following differential equation: $\dot{y}(t) = ay(t) - b$ where both $a, b > 0$. To find the steady state, we solve $\dot{y}(t) = 0 \iff y* = \frac{b}{a}$ and then draw the following diagram:

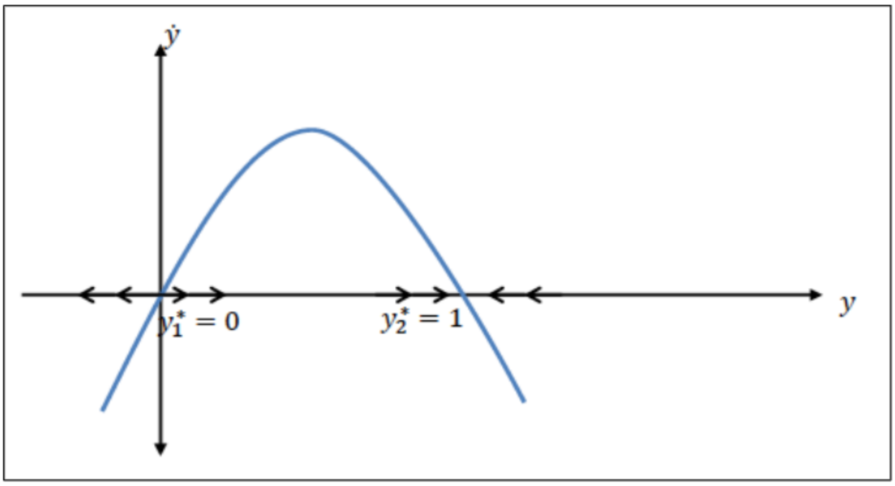
Figure 1: Phase diagram of the equation



Since the slope of $\dot{y}$ is positive, if we are above the steady state level, y will continue to increase and vice versa, implying that any shock will lead the system to never reach the steady system again.

It is also possible for a system to have **multiple steady states**, for example in the case of the differential equation $\dot{y}(t) = y(t) - (y(t))^2$. Setting $\dot{y(t)} = 0$ gives us two steady states ($y_1* = 0$ and $y_2* = 1$) and the following phase diagram:

Figure 2: Phase diagram of the equation



1.4 Economic Applications

What follows is an example of an application of differential equations to a model of supply and demand that takes into account the process of change (seeing that markets are not always at equilibrium).

The Model: Here are the assumptions of the model:

- $Q_d = a - bP$: We have a downward sloping, linear demand function.
- $Q_s = cP$: We have an upward sloping, linear supply function.
- $\dot{P} = \lambda(Q_d(t) - Q_s(t))$: The rate of change of the price of the good will scale by a constant factor λ multiplied by the difference between demand and supply (meaning that if demand is higher than supply, the price will increase).

Using these assumptions, we can deduce the following equations:

$$\dot{P}(t) = \lambda(Q_d(t) - Q_s(t)) = \lambda(a - bP(t) - cP(t)) = \lambda a - \lambda(b + c)P(t)$$

Which is a more complicated version of the non-homogeneous FODE we saw earlier:

$$\dot{P}(t) + \underbrace{\lambda(b + c)}_{u} P(t) = \underbrace{\lambda a}_{w}$$

Letting $\mu := -\lambda(b + c)$ and solving analytically gives us:

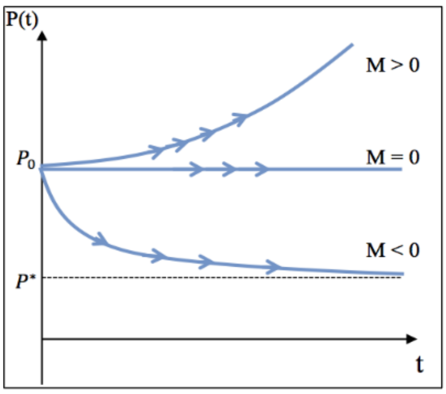
$$P(t) = [P(0) - \frac{a}{b + c}]e^{-\lambda(b+c)t} + \frac{a}{b + c} \implies P(t) = [P(0) - P^*]e^{\mu t} + P^*$$

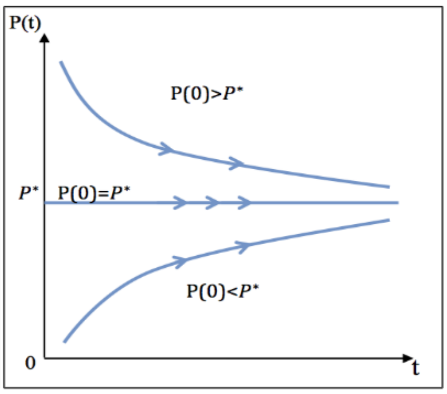
$$\boxed{P(t) = \underbrace{[P(0) - P^*]e^{\mu t}}_{complimentary} + \underbrace{P^*}_{particular}}$$

Here, the **particular solution** gives us what we call **intertemporal equilibrium price** or the **steady state price** which happens to coincide with the solution to the static model. On the other hand, the **complementary solution** is a measure of deviation from the intertemporal equilibrium price.

We can now use what we've derived from the assumptions of our model to see what it predicts in different cases. The figure on the left plots how $P(t)$ would vary across time for different values of μ (M). For example, when it is negative as we initially posited, the gap between $P(t)$ and the equilibrium level will decrease with time. The figure on the right shows how $P(t)$ would change over time for different initial values of $P(0)$ when μ is positive.

Figure 3: Price variation across time





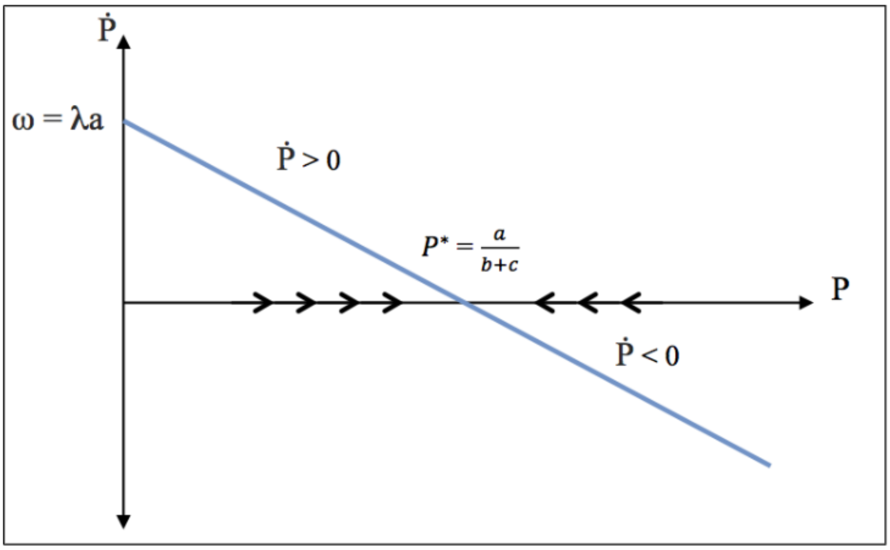
The parameter μ: Taking the derivative with respect to time of $P(t)$ and solving for μ gives us:

$$\mu = \frac{\dot{P(t)}}{P(t) - P^*}$$

meaning that μ is the ratio between the rate of change and the difference between the current price and the equilibrium price, thus determining the **speed of convergence** of the system.

Qualitative solution: If instead we use a phase diagram, we get the following figure that indicates that the system is stable since $\mu < 0$

Figure 4: Phase diagram of price



1.5 Linear approximations

Since we have only learned how to solve linear FODEs, we can potentially deal with non linear FODEs by finding their **linear approximation** using a first order Taylor Series approximation. If $f(x)$ is the function we want to approximate around a, simply use $f(a) + f'(a)(x - a)$

1.6 Growth rates

We are often interested in the **growth rate** of a variable which is given by $\hat{y}(t) = \frac{\dot{y}(t)}{y(t)}$. We can find this by taking the derivative with respect to time of the log of the variable:

$$\frac{\partial ln y(t)}{\partial t} = \frac{1}{y(t)} * \frac{\partial y(t)}{\partial t} = \frac{\dot{y}(t)}{y(t)} = \hat{y}(t)$$

As for the growth rates of variables that are the combination of other variables, we can use the following formulas:

$$y(t) = x(t)z(t) \implies \hat{y}(t) = \hat{x}(t) + \hat{z}(t)$$

$$y(t) = \frac{x(t)}{z(t)} = \hat{x}(t) - \hat{z}(t)$$

The proofs are straightforward applications of the product/quotient rule respectively

2 Introduction

In Macroeconomics, we try to model the complex system that is the world economy. To make the process manageable, we make **simplifying assumptions** that reduce the model to an understandable complexity. Ideally, we want our assumptions to take basis in empirical evidence and to focus on what is important and not unimportant details. It is important to balance the **complexity** of the model with its **realism**

Our models will generally try to focus on a single aspect of the economy, taking other factors as **exogenous** (outside the scope of the model). The goal of the model is to then determine how the **endogenous** aspects of the model react to exogenous changes.

Relative to hard sciences like physics, economics cannot make sure predictions since we are dealing with humans who are necessarily very complex and hard to predict at times. Studying something that is not only alive but also intelligent makes our task that much harder. In addition, we cannot perform experiments for the vast majority of aspects of the economy we'd like to predict and thus we must rely on correlational data most of the time.

Macroeconomics has 2 main areas of study. The first, **growth**, is the study of long term changes in the economy. The second, **cycles**, is the study of short term fluctuations in the economy.

2.1 Demand and Supply

We begin by noting that GDP is the **value of production** or equivalently the sum of all **income** generated. Value of production is generated by the supply of goods while income is determined by the demand for those goods. Thus, both can affect GDP. Does that mean that supply or demand changes are behind changes in GDP?

In the **long run**, it is believed that supply (and thus the constant growth of our capacity to produce) leads to GDP growth. This phenomenon is put forth explicitly by **Say's law** which states that supply creates its own demand. We thus say that in the long run, GDP is **supply determined**.

Nonetheless, there are short term fluctuations in GDP. According to the **classical** school of thought, these changes are due to supply shocks such as bad weather. On the other hand, according to the **Keynesian** school, short term drops are due to firms expecting a drop in demand and decreasing production accordingly, meaning that output would be **demand determined**.

2.2 Models we will study

We will study both a model of economic growth as well as a classical model. The classical model makes the following assumptions:

- Agents are atomistic: No single actor can influence prices.
- Agents are rational: They know what they want and act in their own interest.
- Markets are complete: Participants can insure themselves against contingency.
- Perfect information: All agents have access to perfect information.
- There are no external effects/transaction costs.
- Prices are flexible: Can adjust instantly to changes in demand/supply.
- Markets are perfectly competitive: Outcomes are Pareto efficient

A consequence of this model is that **money is neutral**. This means that real variables are independent of nominal factors such as prices and inflation. For example, doubling money supply would just mean that everything would cost twice as much.

3 Measuring the Economy

Data is essential for economics, allowing the development of theories, the testing of predictions and the guiding of policy. Luckily, a lot of the data that is particularly relevant to economists is readily available such as prices over time and market transactions. To make good use of this data, **National Income accounting** allows for a standardized process of aggregating output that gives us a single measure of overall economic activity, allowing us to make comparisons across countries and time. To do so, we must differentiate between **real** (expressed in terms of goods) and **nominal** (expressed in terms of money) variables.

3.1 GDP

Gross Domestic Product, GDP (Y) is a measure of the **market value** of all **final goods** produced in a country in a given **year**. There are three approaches to measuring it:

- **Value-added**: We want to sum the **value** of all final goods/services produced in the economy. To do so, we add up the value of all goods produced in the economy and subtract intermediate goods (goods used to produce other goods). Make sure to:
  - Goods that are produced but not sold are still considered (as if the firm purchased them)
  - Only count production that occurred this year (if the company sells the unsold goods next year, it doesn't count)
  - Government services are valued at production cost
  - Wages are not considered since they are in a sense intermediate goods used to produce final goods.
- **Expenditure**: We want to sum the **total purchases** of final goods/services. Calculated using $Y = C + I + G + NX$ (consumption, investment, government expenditures and net exports respectively)
  - Investment goods are only final goods purchased by firms that will be used in the long term to produce other goods **capital**
  - Purchases of newly built houses are investments while rent is considered consumption
  - Capital goods bought to replace those affected by depreciation are included in investment. **Gross** measures included depreciation while **net** measures don't.
  - Increases in inventories are considered as investment
- **Income**: We want to sum the **total income** generated in the production of goods/services in the economy. We add up wages and investment income (before taxes, interest payments but after depreciation).
  - People who own their own houses are considered as getting an income equivalent to what it would cost to rent their house. This is to avoid only counting housing services that are actually rented.
  - Transfer payments such as subsidies, interest payments and welfare are not included (just reallocation of income from taxes).

All methods yield almost exactly the same results.

Gross National Product (GNP): As opposed to GDP, GNP also includes the value of production of Canadian firms in **other countries**.

3.2 Nominal vs Real GDP

We use prices to determine the market value of goods/services produced. However, if we want to compare GDP across time, prices can vary as well as quantities produced and thus two equivalently valued levels of production will differ due to inflation. Ideally, we want to measure changes in quantities produced.

This is done by calculating **real GDP** which uses **constant prices** (prices of a base year). We simply multiply the quantities of the current year by the prices of that base year. Thus, in doing so, we make sure that differences are solely due to changes in production quantities.

Finally, to deal with the change of quality of goods over time (ex: computers), we use **hedonic pricing** (the systematic valuation of how much consumers value certain features such as an increase of 1 GHz in processing power).

3.3 Flaws of GDP

We often assume that GDP is also a good indicator of **quality of life** and not just economic activity. However, it does not take into account:

- Unofficial transactions such as the black market.
- The depletion of natural resources for production (should be counted as depreciation).
- The levels of pollution/crime.
- Inequality, education or health

3.4 Price level and Inflation

There are two ways we measure the overall price level of an economy.

GDP Deflator: Since nominal GDP uses current prices and the real GDP uses base year prices, we can take their ratio to determine how much prices have changed since the base year. We thus define the deflator as:

$$Deflator = \frac{Nominal}{Real} * 100$$

Consumer Price Index (CPI): Measures the changes in prices of goods and services **purchased by consumers**. It is calculated by constructing a basket that represents what on average a Canadian household consumes. The price of the basket is calculated for the base year and then calculated again every year whereas the CPI is given by:

$$CPI = \frac{currentbasket}{basebasket} * 100$$

It is the best measure for computing the **cost of living** of an average Canadian. An equivalent measure exists for producers **PPI**.

As for **inflation**, it is defined as the growth rate of the average level of prices ($\hat{P}$). We can compute it using either the CPI or the deflator with the following formula:

$$inflation = \frac{new - old}{old} * 100$$

As usual, there are flaws with these measures of price levels. The CPI does not take into account **changes in buying habits** which arise when one good becomes more expensive, making consumers switch to a less expensive alternative. Additionally, they do not track **changes in quality** or the **introduction of new goods** (both of these produce an upwards bias on inflation).

3.5 Nominal/Real Interest Rates

Interest rates can be considered as **the price of time** (the price to pay to have something today rather than tomorrow). Here we differentiate between the **nominal interest rate** denoted i(t) which is the price at which you can borrow money now to pay back in the future and the **real interest rate** denoted r(t) which is the price at which purchasing power/goods can be borrowed to be paid back in the future. Both of these measures are related to the inflation rate.

The amount of goods we can **buy today** is given by $\frac{1}{P_t}$ and the amount of goods we'll need to **return** in a year is given by $\frac{1+i}{P_{t+1}}$ where the numerator is the amount of money you need to return and the denominator is the price in the next period. Thus, the real interest rate is given by the ratio of the two:

$$1 + r = \frac{\frac{1+i}{P_{t+1}}}{\frac{1}{P_t}} = \frac{1+i}{1+\hat{P}} \implies r \approx 1 - \hat{P}$$

which is the **Fisher equation**. Since we tend to assume that social impatience or how much people would rather have goods today than tomorrow is relatively constant (i.e. the real interest rate is relatively constant). Then changes in the nominal interest rate usually arise from changes in the inflation rate. While the rate charged by the bank is the nominal interest rate, it is the real interest rate which is relevant for expenditure decisions. In general, different types of interest rates (government bonds, corporate bonds, etc.) are closely correlated.

3.6 Unemployment

Unemployment is accounted for in the following manner. Let L, N, U and u bet the labor force, the number of employed people, the number of unemployed people and the unemployment rate respectively. We consider someone **unemployed** if they have no job, want to work and are actively looking for a job. The labor force is equal to the sum of the unemployed and the employed. The unemployment rate is the number of employed over the labor force:

$$L = U + E, u = \frac{U}{L}$$

Types of Unemployment

- **Frictional**: Unemployment arising from the process of finding and changing jobs. It is normal and healthy to have some amount of frictional unemployment
- **Structural**: Unemployment arising from long term economic changes in industries leading to whole sectors becoming unemployed.
- **Cyclical**: Unemployment arising from changes in the business cycle (recessions).

3.7 International Trade

We can divide a country's transactions with the rest of the world into 2 components: the **trade account balance** and the **capital account balance**.

The trade account balance is simply **net exports** (exports - imports). If the balance of the account is positive, there is a **trade surplus**, otherwise it is a **trade deficit**.

The capital account balance is the difference between the sale of Canadian assets (**capital inflows**) and the purchase of assets from the rest of the world (**capital outflows**). If it is positive, Canada is **borrowing** (selling more than it is buying) and if it is negative Canada is **lending**.

Combining these two accounts gives us the **balance of payments** which is equal to the sum of both accounts and **must be zero**. This means that if a country has a trade surplus, it must be buying more assets than it is selling and if it has a trade deficit it is selling more assets than it is buying (to finance its imports).

Being open on financial markets allows countries to run surpluses and deficits. Nonetheless, the vast majority of international transactions are related to assets and not necessarily goods.

3.7.1 Prices

When considering an open economy, we must also consider the **nominal exchange rate** (e) which is the price of foreign currency relative to domestic currency. Increases in this exchange rate indicate either a depreciation of the domestic currency or an appreciation of the foreign currency).

We must also consider the price level of foreign countries relative to our own. This information is given by the **real exchange rate** (ϵ) which is calculated using:

$$\epsilon = e\frac{P*}{P} = \frac{CAD}{EUR} * \frac{EUR/EUgood}{CAD/CAgood} = \frac{CAgood}{EUgood}$$

where $P*$ is the foreign price level. An increase of the real exchange rate indicates that foreign goods are becoming less expensive relative to domestic goods.

3.7.2 The equilibrium nominal exchange rate

The nominal exchange rate is determined by the supply and demand of goods and the correct level can be calculated in 2 ways.

The first one involves the **trade account** and is based on the **non-arbitrage condition for goods** which states that the price of a Canadian good should be equal to the price of that same good in the US multiplied by the exchange rate:

$$P_{CA} = P_{US} * e$$

If this condition is not met it is then theoretically possible to perform "arbitrage" by either buying the good in Canada, selling it for USD and then converting the USD to CAD or buy the good in the US, sell it in Canada and get more USD back depending on the direction of the inequality.

While there are barriers to this such as import fees and transportation fees, the general notion remains that if $P_{CA} < P_{US} * e$, then there will be an increase in demand for CAD which will increase the exchange rate and bring it closer to equilibrium and vice versa.

If we log differentiate both sides of the equation to examine the growth rate of both sides, we get:

$$\hat{P}_{CA} = \hat{P}_{US} + \hat{e} \iff \hat{e} = \hat{P}_{CA} - \hat{P}_{US}$$

Meaning that if Canadian inflation is higher than US inflation, then the Canadian dollar will depreciate.

The second method involves the **capital account** and is based on **uncovered interest-rate parity** which states that the **expected rates of return** in 2 countries, expressed in a common currency should be the same. If not, investors will buy assets in the country with the higher expected return.

$$i_{CA} = i_{US} + E[\hat{e}] \iff E[\hat{e}] = i_{CA} - i_{US}$$

If the interest rate in Canada is higher, then investors expect the Canadian dollar to depreciate.

3.8 The saving-investment identity

If we use the expenditure approach of GDP we have GDP as:

$$Y = C + I + G + (X - M)$$

We now consider the notion of **disposable income** which is simply income after taxes. Since household income is equal to nation product (all income from production goes to households before taxes), we have that $Y_d = Y - T$. Households can then either consume (C) or save (S) their income which gives us the following identity:

$$Y_d = Y - T = C + S \implies Y = C + S + T$$

Plugging back into the expenditure equation gives us:

$$C + S + T = C + I + G + (X - M) \implies \underbrace{S + \underbrace{(T - G)}_{\text{Public Saving}}}_{\text{Domestic Savings}} + \underbrace{(M - X)}_{\text{Foreign Saving}} = I$$

Which implies that **total savings are equal to investment**. The identity can be used to analyze trade deficits. When domestic saving decreases relative to investment, then the balance must be made up by trade deficits (foreign savings). Similarly, when governments run deficits while household savings/total investment remain constant, then foreign savings and thus the trade deficit will increase to pay for the government deficit.

The only way to not increase the trade deficit while running a government deficit is to let investment fall.

4 Growth

Having a model for economic growth is essential for allowing us to understand the economic evolution of various countries and the discrepancies that exist between them.

4.1 Comparisons between countries

To make meaningful comparisons between countries, We need to express both their GDPs in a common currency and then make adjustments based on their relative price levels. The general formula to make the GDP of country 2 comparable to the GDP of country 1 is given by (where both are already in per capita terms):

$$GDP_2 * e * \frac{P_1}{P_2}$$

where e is the exchange rate from country 2 dollars to country 1 dollars. This adjustment is referred to as **Purchasing Power Parity Adjusted**.

Due to the **compounding effects** of growth, small changes in growth rates lead to large differences in per capita outcome over time. In fact, these effects tend to largely **outweigh small variations** that occur due to business cycles (e.g. an economy that goes through a recession will still have a significantly higher per capita GDP than it did 50 years ago assuming it had a decent growth rate during that period).

Additionally, GDP per capita is **strongly correlated** with **multiple measures of well-being** such as life expectancy, access to water as well as reductions in infant mortality, malnutrition and poverty rates. The causal effects likely go both ways as healthier people are more productive also.

Finally, even with growth is sometimes accompanied by **increasing inequality**, there is strong evidence that suggests that it makes everyone better off and is one of the best ways of **reducing poverty**

4.2 Production Functions

Instead of having a separate production function for each firm, in Macro we define an **aggregate production function** that includes all firms and produces a **composite good** (GDP). To produce GDP, we assume only 2 inputs are needed: **capital (K)** and **labor (N)** with both inputs being owned by households and thus the production function being given by:

$$Y = F(K, N)$$

Properties of the production function: Here are the properties that we want our production function to have:

- Increasing input will always increase output. This implies that our production function **increases monotonically**.
- If we hold one input constant while increasing the others, the additional amount of output produced diminishes. This means that there are **diminishing marginal products** for labor and capital. While the first property implies that the marginal product is always **positive**, it is also always **decreasing**.
- Doubling labor and capital will double output. This property implies **constant returns to scale** or mathematically $F(cK, cN) = cF(K, N)$.

Combining these properties gives us that the two inputs are **complementary** in production since increasing one increases the marginal product of the other. Additionally, the **size of firms** is irrelevant due to constant returns to scale. We can then use a **representative firm** which we consider as **competitive** meaning that it cannot influence prices.

Additionally, **constant returns to scale** imply that the production function is **homogeneous of degree one** ($F_K(K,N) = F_K(\lambda K, \lambda N), \forall k > 1$). Thus, Euler's theorem tells us that:

$$Y = F_K * K + F_N * N$$

where F_K and F_N are their respective marginal products. Under the assumption of competitiveness, each input is paid their **marginal product** and thus all of output is divided into payments to labor and capital (there are K units of labor paid their marginal product and similarly for labor).

Cobb-Douglas: No Economics course would be complete without the use of a Cobb-Douglas production function and thus it will be the one we are using:

$$Y = K^{\alpha} N^{1-\alpha}$$

where α denotes the importance of capital and $(1-\alpha)$ the importance of labor. If we derive their respective marginal products we get:

$$MPN = (1-\alpha)(\frac{K}{N})^{\alpha} = (1-\alpha)\frac{Y}{N}$$

$$MPK = \alpha(\frac{N}{K})^{1-\alpha} = \alpha\frac{Y}{K}$$

Both of these marginal products are positive which satisfies the condition that production function are **increasing in each input**. Additionally, if we take the partial derivative of MPN relative to labor once again we get $-\alpha(1-\alpha)K^{\alpha}N^{-\alpha-1} < 0$ which implies that there are diminishing marginal products as we increase labor while keeping capital constant (respects property 2).

Finally, Cobb-Douglas respects **constant returns to scale** (just plug in an arbitrary constant to check) and taking the partial derivative relative to labor then capital gives $\alpha(1-\alpha)k^{\alpha-1}n^{alpha} > 0$ meaning that the factors are **complementary** (increasing one, increases the marginal product of the other). We have thus found a production function that respects all properties we wanted.

4.3 Kaldor's Stylized Facts

First we establish the following notation:

- Π: Aggregate accounting **profits** (payments to capital)
- W: Aggregate **wages** (payments to labor)
- r: Return to capital (how much each unit of capital earns)
- w: Wage rate (wage per unit of labor)
- n: Growth rate of population/labor force (assumed to be roughly constant and positive.

We thus have the following relations:

$$w = \frac{W}{N}, r = \frac{\Pi}{K}$$

Since Y is aggregate production which, when calculated using the **income method** is simply the sum of income from wages and income from capital, we have that $Y = W + \Pi$. We can then find their respective shares of total income as:

- **Capital income share**: Share of income that goes to labor $\frac{W}{Y}$
- **Labor income share**: Share of income that goes to capital $\frac{\Pi}{Y}$

Finally, **investment** is simply the rate of change of capital and thus $\dot{K} = I$ and thus the **saving rate** (equivalent to the investment rate) is simply $s = \frac{\dot{K}}{Y}$.

We can now state Kaldor's stylized facts which are regularities observed when examining economies over large periods of times.

1. Per capita output grows at a **relatively constant, positive rate**. $\dot{Y}_i >= c > 0$.
2. The **ratio** between **capital and output** does not have a trend. $(\frac{\hat{K}_i}{Y_i}) = 0$
3. The **return to capital** shows no trend. $\hat{r} = 0$
4. **Income per capita** and **growth rates** vary tremendously from country to country.

Consequences of stylized facts: From Kaldor's stylized facts, we can derive the following properties:

1. From 2. and 3., we have that $\hat{K}_i$ must be roughly equal to $\hat{Y}_i$ for the ratio not to change and thus $\hat{K}_i = c > 0$
2. Since $\hat{K} = \hat{K}_i + n$, both of which are constant, $\hat{K} = c > 0$
3. $\hat{K} = \frac{I}{K} = \frac{I}{Y}\frac{Y}{K}$. Since $\frac{Y}{K}$ is constant by 2. and we have just shown $\hat{K}$ is constant, then $\frac{I}{Y} = s$ must also be constant
4. $r = \frac{\Pi}{K} = \frac{\Pi}{Y}\frac{Y}{K}$. Since $\frac{Y}{K}$ and r are constant, we have that $\frac{\Pi}{K}$ is also constant (the KIS is constant).
5. Since the KIS is constant and LIS is 1-KIS, we have that the LIS is constant.
6. $w = \frac{W}{N} = \frac{W}{Y}\frac{Y}{N} \implies \hat{w} = 0 + (\frac{\hat{Y}}{N}) = \hat{Y}_i > 0$. Thus wage grows at a positive, stable rate.

With this, we are now ready to **model growth**. We will be using **supply-side models**, i.e. models where **Say's law** applies meaning that supply generates its own demand.

5 The Solow Growth Model

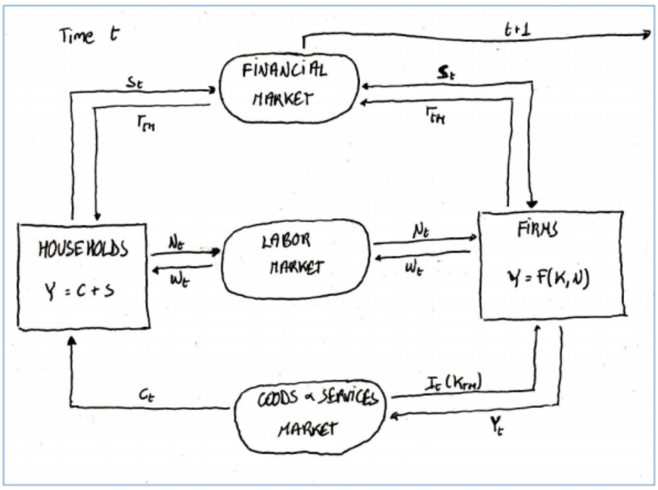
Capital per worker is highly correlated to **GDP per worker**, suggesting that understanding the evolution of capital per worker is essential for understanding growth of income capita. The **Solow model** attempts to model growth as the process of **capital accumulation**.

5.1 Model without technological change

We begin by describing the general structure of the model and the assumptions we make.

5.1.1 Structure of the model

Figure 5: Diagram of the relations between components of the model



Assumptions

- Markets are **competitive** and households/firms take **prices as given**
- Only output, capital and labor are transacted
- Money is **neutral** and prices are **flexible**
  - The only prices are the **real interest rate** and the **real wage rate** (expressed in units of output)
- **Say's law** holds and supply creates its own demand
- The economy is **closed** and we don't consider the **government**
- Population growth (n) is **exogenous** and evolves at a constant, exponential rate.
- Capital **depreciates** at a constant rate δ.

Production Function: Once again we use the **Cobb-Douglas** production function

Market Structure: To understand the structure of the market, we need to find equations for **supply and demand** of both capital and labor. Using the expenditure decomposition of national product we have:

$$Y(t) = C(t) + I(t) \implies I(t) = Y(t) - C(t)$$

Then, using that consumption is simply total income times $1 - s$, we have:

$$C(t) = (1 - s)Y(t) \implies I(t) = Y(t) - (1 - s)Y(t) = sY(t)$$

Since the rate of change of capital is just the amount added from investment - the amount lost from depreciation:

$$\dot{K}(t) = I(t) - \delta K(t) = sY(t) - \delta K(t)$$

- **Supply**: Supply of both inputs is determined by its ability to be produced since this model is **supply-sided**
  - Each households supplies one unit of labor inelastically (supplied regardless of wage) and thus labor supply is given by $N(t)$
  - Households also own the capital that they rent to firms and its evolution is given by $sY(t) - \delta K(t)$
- **Demand**: To determine the demand of firms for capital and labor, we must solve the **profit maximization problem**:

$$\max_{N(t),K(t)} K(t)^{\alpha} N(t)^{1-\alpha} - w(t)N(t) - (r(t) + \delta)K(t)$$

  where we normalize the price of output to 1. Taking each partial derivative and setting them equal to 0 yields the following first order conditions
  - $N(t) = (\frac{1-\alpha}{w(t)})^{\frac{1}{\alpha}} K(t)$
  - $K(t) = (\frac{\alpha}{r(t)+\delta})^{\frac{1}{1-\alpha}} N(t)$

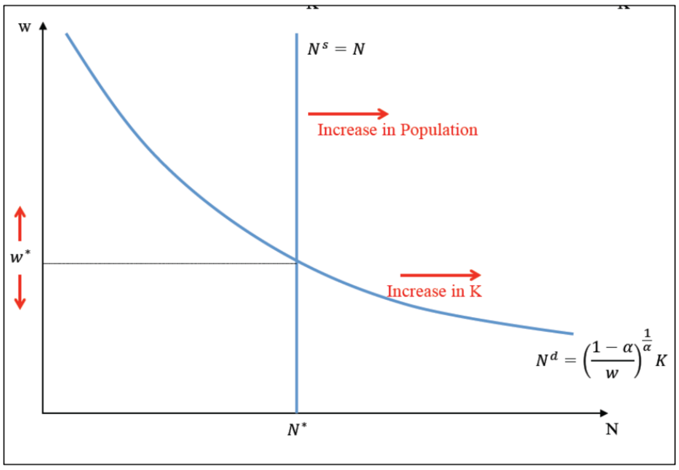
Now we equate supply and demand to obtain the **equilibrium wage and interest rate**:

$$w^* = (1 - \alpha)(K^*)^{\alpha} (N^*)^{-\alpha}$$

$$r^* + \delta = \alpha (K^*)^{\alpha-1} (N^*)^{1-\alpha}$$

Using this information, we can draw the graph of the **labor market**. Under these conditions, using Euler's Theorem, we see that firms **make no economic profit**

Figure 6: Diagram of the labor market



5.1.2 Evolution of the model

Equilibrium: Seein ghat the Solow model is dynamic, we do not have a single equilibrium but rather a **sequence of equilibria**. At each period, K(t) and N(t) vary which then affect all the other variables. Here is a summary of the equations that determine them.

$$N(t) = N_0 e^{nt}$$

$$\dot{K}(t) = sY(t) - \delta K(t)$$
$$Y(t) = K^\alpha N^{1-\alpha}$$
$$C(t) = (1-s)Y(t)$$
$$w(t) = (1-\alpha)\,(K^*)^\alpha\,(N^*)^{-\alpha}$$
$$r + \delta = \alpha\,(K^*)^{\alpha-1}\,(N^*)^{1-\alpha}$$

Per Capita Variables: Since our model seeks to explain per-capita changes, we rewrite some of our variables in per capita terms:

$$Y = K^\alpha N^{1-\alpha} = \left(\frac{K}{N}\right)^\alpha N = NK_i^\alpha \implies Y_i = K_i^\alpha$$

$$K_i = \frac{K}{N} \longrightarrow \dot{K}_i = \frac{\dot{K}N - \dot{N}K}{N^2} = \frac{\dot{K}}{N} - nK_i = \frac{sY - \delta K}{N} - nK_i = sY_i - (\delta+n)K_i$$
$$w = (1-\alpha)(K)^\alpha(N)^{-\alpha} = (1-\alpha)\,(K_i)^\alpha$$
$$r + \delta = \alpha(K)^{\alpha-1}(N)^{1-\alpha} = \alpha\,(K_i)^{\alpha-1}$$

Steady State The Solow model reaches a steady state when **capital per capita** is constant. When this steady state is reached is determined by the **fundamental equation of the Solow model**:

$$\boxed{\dot{K}_i = sK_i^\alpha - (\delta+n)K_i}$$

This equation states that the **rate of change of capital** is simply the difference between **investment** and **replacement investment** (the amount of investment needed to maintain the same level of capital per capita given depreciation and population growth.

We can solve this **analytically** by setting $\dot{K}_i = 0$:

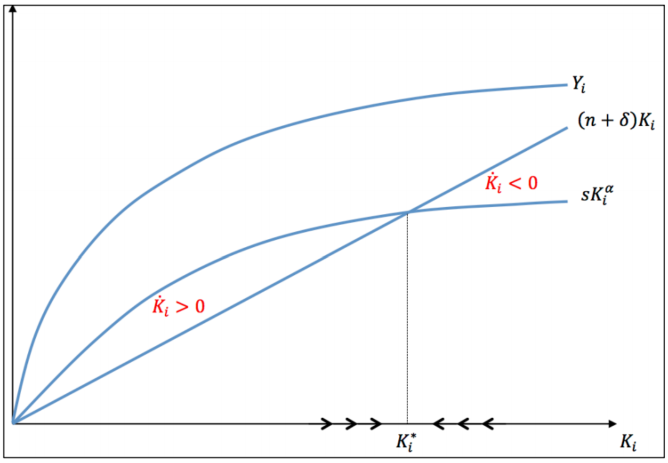
$$K_i^* = (\frac{s}{\delta+n})^{\frac{1}{1-\alpha}}$$

$$Y_i^* = (K_i^*)^\alpha = (\frac{s}{\delta+n})^{\frac{\alpha}{1-\alpha}}$$

Thus, the results of our model indicate the **savings rate** and **population growth** are the main factors explaining differences in per capita output.

Additionally, our model respects most of the stylized facts since the **capital-income ratio** and the **real interest** are both **constant** once the steady state is reached. However, it does not match the prediction of **constant growth of income per capita**.

Figure 7: Qualitative steady state of the Solow model



5.2 Model with technological change

5.2.1 Structure of the model

To allow for **growth in output per capita** in our model, we need to include the notion of **labor-augmenting technological progress** (also known as **total factor productivity**) which modifies our production function to:

$$Y(t) = K(t)^\alpha (A(t)N(t))^{1-\alpha}$$

What this technological progress models is the change in **effective labor** given by AN, meaning that workers become more effective (can produce more with the same level of capital) as technology improves. This new production function maintains constant returns to scale (unless we increase technology as well).

We assume that **TFP** is **non-rival** and **non-excludable** which means it is **pure public good**. Once it becomes available, all firms can use it without hampering the ability of other firms to use it.

We now rewrite the production function in per capita terms yielding:

$$Y_i = K_i^\alpha A^{1-\alpha}$$

Which is the per capita production function we will be working with from now on.

5.2.2 Evolution of the model

Now that we've established the general structure of the model, we examine its evolution through time. We have the following **laws of motions**:

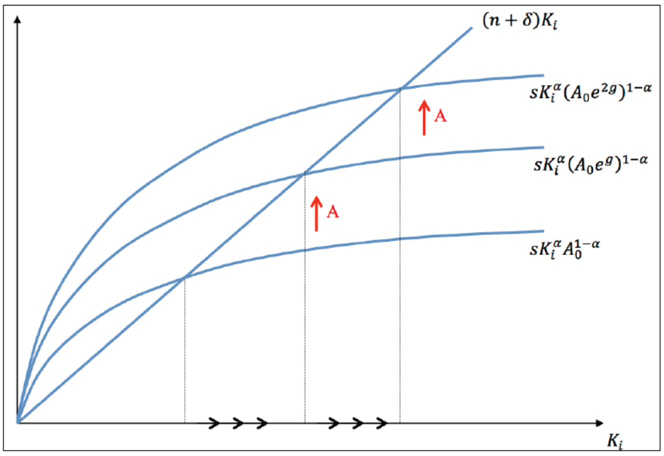
$$N(t) = N_0 e^{nt}$$

$$\dot{K} = sY - \delta K$$

$$\hat{A} = g \implies A(t) = A_0 e^{gt}$$

$$\dot{K}_i = sY_i - (\delta + n)K_i = sK_i^\alpha A^{1-\alpha} - (\delta + n)K_i$$

Figure 8: The effects of increasing technological change on the Solow model



As illustrated qualitatively, this model **does not reach a steady state** since TFP is constantly increasing.

Steady State: Instead of looking at a steady state where the **level of output per capita** is constant, we instead look for one where its **growth rate** is constant. To determine the growth rate of output per capita, we log-differentiate it:

$$\hat{Y}_i = \alpha\hat{K}_i + (1-\alpha)\hat{A} = \alpha\hat{K}_i + (1-\alpha)g$$

For this to be a constant term,we need the growth rate of capital to be constant as well

$$\dot{K}_i = sY_i - (\delta+n)K_i \implies \hat{K}_i = s\frac{Y_i}{K_i} - (\delta+n)$$

which is only constant if the **output/captial ratio is constant**. For this ratio to be constant, we need both the growth rate of capital and output to be the same (denote by k). If we combine this with our initial equation on the growth rate of output, we get:

$$k = \alpha k + (1-\alpha)g \implies (1-\alpha)k = (1-\alpha)g \implies g = k$$

We call this **dynamic steady state** the **stable growth path**. This constant growth rate requires our per capita variables to grow at the same rate as technology. By rewriting these variables in units of effective labor, we ensure that as technology improves, they however remain constant:

$$y = \frac{Y_i}{A}, k = \frac{K_i}{A}$$

Our production function then becomes:

$$y = k^\alpha$$

and the law of motion of capital:

$$\dot{k_i} = sy - (\delta+n+g)k$$

Combining these two gives us the **fundamental equation of the Solow model (with technological change**:

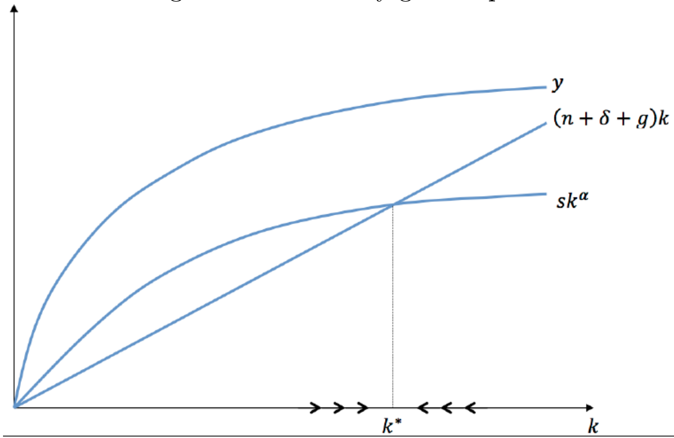
$$\boxed{\dot{k_i} = sk^\alpha - (\delta+n+g)k}$$

Solved qualitatively, we get a similar phase diagram with a similar intuition behind it. Calculating the steady state levels of output and capital in the same manner as before yields

$$k^* = \left(\frac{s}{n+\delta+g}\right)^{1/1-\alpha}$$

$$y^* = \left(\frac{s}{n+\delta+g}\right)^{\alpha/1-\alpha}$$

Figure 9: The steady growth path



As for the growth rate of output along the **stable growth path**, it is given by g and the growth rate of output per capita is given by $n + g$.

Ultimately this model indicates that changes in **TFP**, **savings rates** or **population growth** are all behind differences in income per capita.

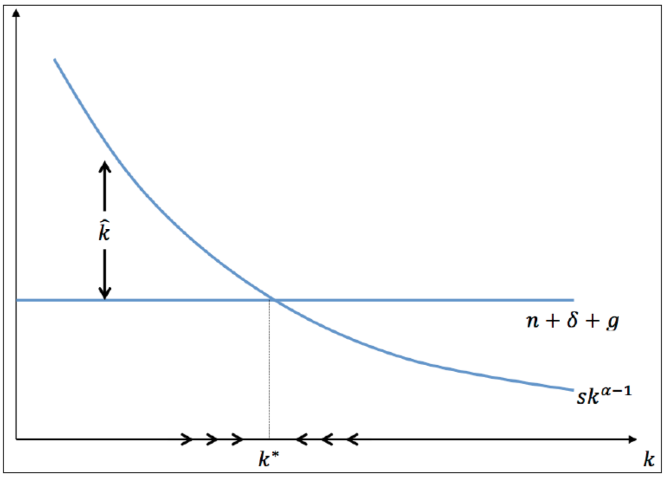
5.2.3 Transitions

Dividing the basic equation of the model by k yields:

$$\hat{k} = sk^{\alpha-1} - (\delta + n + g)$$

which implies that low levels of capital are associated to high growth rates of capital (and thus output). Qualitatively, it looks like so:

Figure 10: Diminishing marginal product of capital



As for analytically, we need to linearize $\dot{k}$ around k^*:

$$\dot{k} \approx [s\alpha(k^*)^{\alpha-1} - (g + n + \delta)](k - k^*) = \lambda(k - k^*) \implies k(t) = k^* + e^{\lambda t}(k(0) - k^*)$$

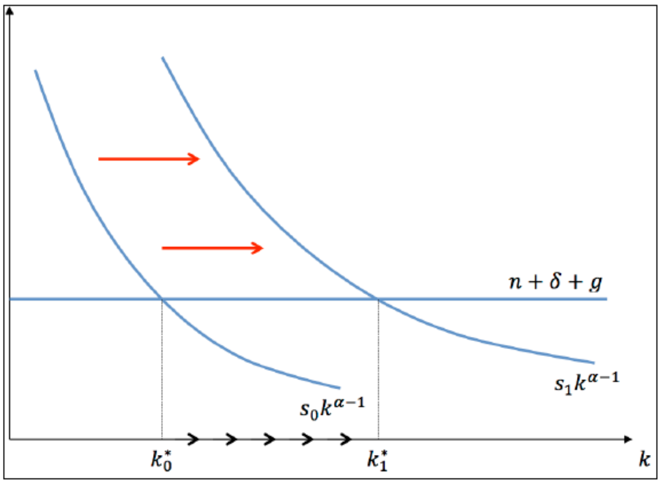
where $\lambda = [s\alpha(k^*)^{\alpha-1} - (g + n + \delta)]$. We can then rewrite the eigenvalue as

$$\lambda = -(1 - \alpha)(g + n + \delta) < 0$$

which implies that the steady state is stable.

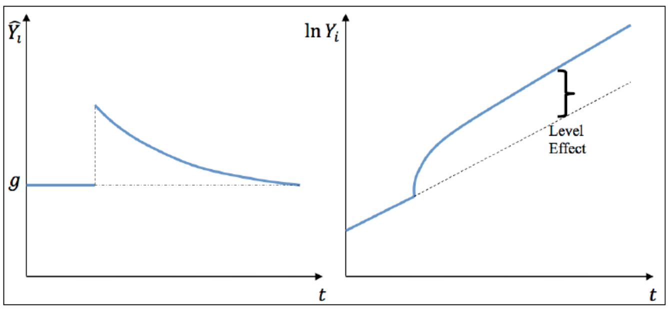
While changes in key parameters of the model lead to temporary growth effects, they are not permanent and eventually a new steady state is reached. Nonetheless, there is a permanent level effect meaning total output is permanently higher then if there hadn't been the change. **Properties of the model**:

Figure 11: Effect of a change in savings rate



- The economy will always eventually achieve a steady state (low levels of capital imply high MPK which leads to high investment which results in increasing capital towards the steady state). On the other hand, high levels of capital lead to low MPK which leads to low investment which results in capital not keeping up with replacement investment.
- The further an economy is below its steady state, the fast its rate of growth is.
- There are 2 types of changes, technological changes which lead to permanent growth of growth rates and transitional changes which only have level effects (such as policy changes).

Figure 12: Transitional dynamics



5.3 Golden Rule Level of Capital

Since consumption is a fairly accurate indicator of welfare, we consider its steady state level as well as the level of saving that maximizes steady state consumption which is the **golden rule level of saving** and is reached by setting $s = \alpha$.

As for the golden rule level of capital, it is given by:

$$MPK = \alpha(k)^{\alpha-1} = (n + \delta + g)$$

or basically the **marginal product of capital** should be equal to the sum of population growth, technological change and depreciation (**slope of replacement investment line**).

5.4 Growth Accounting

Growth accounting is the process by which we attempt to approximate the **impact** of the **various inputs** of our model to growth. We begin with a production function with neutral technical change:

$$Y = AK^{\alpha}N^{1-\alpha}$$

Log differentiation gives us the growth rate:

$$\hat{Y} = \hat{A} + \alpha\hat{K} + (1-\alpha)\hat{N}$$

Since the growth rate of per capita GDP is $\hat{Y} - \hat{N}$

$$\hat{Y_i} = \hat{A} + \alpha\hat{K} + -\alpha)\hat{N} = \hat{A} + \alpha\hat{K}_i$$

And thus we can decompose the growth rate of per capita GDP into the growth rate of technology and the growth rate of capital per capita. Rewriting the expression gives us the **Solow residual** which is equivalent to **TFP**:

$$\hat{A} = \hat{Y_i} - \alpha\hat{K}_i$$

Then, by comparing the **growth rate of output** to the **growth rate of TFP** and the **growth rate of K_i** we can determine how much each contributes to growth.

We then notice that a majority of growth seems to be driven by growth in **TFP**. More complicated models also include the impact of changes such as differences in **sectoral employment** (agriculture tends to have lower output per worker than industry) or **changes in education**.

Ultimately, the Solow Model indicates that we **cannot count on the accumulation of capital** in the long term to drive growth. Instead, **long term growth** is driven by **technological change.**